

Classical closure theory and Lam's interpretation of ϵ -RNG¹

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Abstract

Lam's phenomenological ϵ -renormalization group (RNG) model is quite different from the other members of that group. It does not make use of the correspondence principle and the ϵ -expansion procedure. In this report, we demonstrate that Lam's ϵ -RNG model [*Phys. Fluids A*, **4**, 1007 (1992)] is essentially the physical space version of the classical closure theory [Leslie and Quarini, *J. Fluid Mech.*, **91**, 65 (1979)] in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

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Introduction

In this note, we demonstrate that Lam's ϵ -RNG model¹ is essentially the physical space version of the classical closure theory² in spectral space and consider the corresponding treatment of the eddy viscosity and energy backscatter.

Analysis

The incompressible N-S equations are

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu_0 \nabla^2 \mathbf{v} \quad (1)$$

where ν_0 is the molecular viscosity, ρ is the density, and p is the pressure and can be determined from (1) using $\nabla \cdot \mathbf{v} = 0$. The external driving force that sustains the turbulence and which acts in the very small wavenumber region is not included in (1) since it plays no part in the energy cascade process in the inertial range².

As in both closure and RNG theories, the velocity field is filtered into two components

$$\mathbf{v} = \mathbf{v}^< + \mathbf{v}^>, \quad p = p^< + p^> \quad (2)$$

where the Fourier-transformed fields

$$v_i^<(\mathbf{k}, t) = G(k) v_i(\mathbf{k}, t), \quad (3)$$

$$v_i^>(\mathbf{k}, t) = (1 - G(k)) v_i(\mathbf{k}, t). \quad (4)$$

The sharp cut-off filter of classical closure theory is exactly the same as the RNG technique of separating the subgrid from the resolvable scales at the cutoff wavenumber Λ

$$G(k) = \begin{cases} 0 & \text{if } k > \Lambda ; \\ 1 & \text{if } k < \Lambda. \end{cases} \quad (4)$$

In the classical closure theory of Leslie and Quarini (LQ)², the filtered N-S equation is

$$\left(\frac{\partial}{\partial t} + [\nu_0 + \nu_E(k)] k^2 \right) v_\alpha^<(\mathbf{k}, t) = M_{\alpha\beta\gamma}(k) \int d\mathbf{p} d\mathbf{q} v_\beta^<(\mathbf{p}, t) v_\gamma^<(\mathbf{q}, t) + f_\alpha(\mathbf{k}, t), \quad (5)$$

where $M_{\alpha\beta\gamma}(k)$ is the standard nonlinear coupling coefficient^{2,3}. For convenience we have added to *both* sides a wavenumber dependent turbulent eddy viscosity $\nu_E(k)$, which is at the moment unspecified. The term $f(k, t)$ accounts for the Reynolds stress^{2,4},

$$R_{\beta\gamma} \equiv v_{\beta}^{>}(\mathbf{p}, t)v_{\gamma}^{>}(\mathbf{q}, t). \quad (6)$$

the cross stress^{2,4},

$$C_{\beta\gamma} \equiv v_{\beta}^{<}(\mathbf{p}, t)v_{\gamma}^{>}(\mathbf{q}, t) + v_{\beta}^{>}(\mathbf{p}, t)v_{\gamma}^{<}(\mathbf{q}, t) \quad (7)$$

and the added eddy viscosity $\nu_E(k)$:

$$f_{\alpha}(\mathbf{k}, t) \equiv \nu_E(k)k^2 v_{\alpha}^{<}(\mathbf{k}, t) + M_{\alpha\beta\gamma}(k) \int d\mathbf{p}d\mathbf{q} [C_{\beta\gamma} + R_{\beta\gamma}]. \quad (8)$$

In (6)-(7), $|\mathbf{p} + \mathbf{q}| < \Lambda$. It is important to realize that no random force has been inserted here.

In the Lam approach to ϵ -RNG¹, one works in physical space rather than wavenumber space. The exact resolvable scale Navier-Stokes equations can be written

$$\left[\frac{\partial}{\partial t} - (\nu_0 + \nu_T)\nabla^2\right]\mathbf{v}^{<} = -\frac{1}{\rho}\nabla p^{<} - \nabla \cdot (\mathbf{v}^{<}\mathbf{v}^{<}) + \mathbf{g}^{fast} \quad (9)$$

where \mathbf{g}^{fast} is defined by

$$\mathbf{g}^{fast} = \nabla \cdot (\mathbf{v}^{<}\mathbf{v}^{<} - \mathbf{v}\mathbf{v}) - \nu_T\nabla^2\mathbf{v}^{<} = \nabla \cdot (2\mathbf{v}^{>}\mathbf{v}^{<} - \mathbf{v}^{>}\mathbf{v}^{>}) - \nu_T\nabla^2\mathbf{v}. \quad (10)$$

Note that Lam has introduced a *k-independent* turbulent eddy viscosity, ν_T , which remains to be chosen. \mathbf{g}^{fast} is generated by the filtering process. The term \mathbf{g}^{fast} in physical space corresponds to the term $\mathbf{f}(\mathbf{k}, t)$ in wavenumber space, in Eq. (8).

The classical theory proceeds from this point by the use of certain “closure approximations”^{2,3}. An equation for the resolvable spectral energy, $\bar{E}(k, t)$, can readily be derived,

$$\left[\frac{\partial}{\partial t} + 2\nu_0 k^2\right]\bar{E}(k, t) = \bar{T}(k, t) + T^{>}(k, t), \quad (11)$$

where $\bar{T}(k, t)$ is the resolvable scale energy transfer and $T^>(k, t)$ is the energy transfer caused by the cross and Reynolds stresses² which can be put into the form^{2,5}

$$T^>(k, t) \equiv -2\nu_d(k)k^2\bar{E}(k, t) + U(k). \quad (12)$$

$U(k)$, which represents the backscatter of energy from small to resolvable scales and is also the spectrum of the correlation function of \mathbf{f} , is given by

$$U(k) \equiv \int_{\Delta} dp dq B(k, p, q) E(p) E(q) G^2(k) [1 - G(p)G(q)]. \quad (13)$$

$\nu_d(k, t)$, the *drain eddy viscosity*, is given by

$$\nu_d(k) \equiv \int_{\Delta} dp dq A(k, p, q) E(q) [1 - G(p)G(q)]. \quad (14)$$

The integration domain is denoted by the expression Δ in which p and/or $q > \Lambda$. The explicit functional forms of A and B appearing in (13)-(14) are given in Leslie³ and LQ².

Instead of trying to compute \mathbf{g}^{fast} using closure approximations, Lam¹ simply tries to model its correlation function based on physical arguments. In his view, \mathbf{f} is simply a guess of what \mathbf{g}^{fast} should be for $k \approx \Lambda$ in the resolvable scale Navier-Stokes equation. He noted that in the absence of \mathbf{f} , the energy spectrum of the flow, computed from (5) driven by initial and/or boundary conditions, will have a Kolmogorov dissipation wavenumber substantially smaller than Λ . The primary role of \mathbf{f} is to extend for the resolvable scale velocity field the inertial range with a guaranteed Kolmogorov scaling for $k \approx \Lambda$ and beyond.

The forcing function in classical closure theory arises from filtering at the small scales. In modeling the correlation function of \mathbf{f} , Lam¹ *assumes* the form

$$\langle f_i(\mathbf{k}, \omega) f_j(\mathbf{k}', \omega') \rangle = \frac{2}{\Pi_3} \mathcal{E} \frac{1}{\Lambda^{4-\epsilon}} k^{-d+4-\epsilon} (2\pi)^{d+1} P_{ij}(k) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \quad (15)$$

where ω is frequency, \mathcal{E} is the dissipation rate, d is the dimension of the physical space, Π_3 is a constant, and $P_{ij}(k) = \delta_{ij} - k_i k_j / k^2$. A multiplicative factor involving $\Lambda^{4-\epsilon}$ is introduced

to maintain dimensional consistency for arbitrary ϵ . It is of some interest to compare Eq. (15) with the forcing correlation function introduced by Yakhot and Orszag (YO)⁶

$$\langle f_i(\mathbf{k}, \omega) f_j(\mathbf{k}', \omega') \rangle = \frac{2}{\Theta} \mathcal{E} k^{-d+4-\epsilon} (2\pi)^{d+1} P_{ij}(k) \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'), \quad (16)$$

where Θ is a known constant determined by $2D_0 S_d / (2\pi)^{d+1} = 1.594\mathcal{E}$ (YO⁶) and S_d is the area of a d -dimensional unit sphere. This form⁷ is assumed to arise from forcing at $k = 0$:

$$\langle ff \rangle = \delta(k) \mathcal{E} \delta(\mathbf{k} + \mathbf{k}') \quad (17)$$

with the use of Gel'fand's δ -function representation in the limit of $\epsilon \rightarrow 4$ and $k \rightarrow 0$

$$\delta(k) = \lim_{\epsilon \rightarrow 4} (4 - \epsilon) k^{1-\epsilon}. \text{ for } k \rightarrow 0 \quad (18)$$

To recover (16), it appears that (18) needs to be applied for $k \neq 0$, without the $(4 - \epsilon)$ factor.

Lam pointed out that the forcing correlation function, Eq. (15), should peak around Λ ; that its magnitude should be small for small k by an appropriate choice of ν_T ; and that its behavior for $k \gg \Lambda$ is unimportant and irrelevant for the evolution of the resolved modes. Most importantly, the correlation function now depends on Λ , while in ϵ -RNG⁶⁻⁷, the correlation function is assumed to be “scale invariant”. The dimensionless parameter ϵ in the correlation function is now available as a freely adjustable parameter, and Lam used it to make the “predicted value” of Kolmogorov constant acceptable. He showed that either $\epsilon = 0$ or $\epsilon = 0.923$ yield good results.

The stochastic backscatter \mathbf{f} , for isotropic homogeneous turbulence in three dimensions, has a k^4 spectrum to lowest order in wavenumber k (e.g., Ref. 5). Specifically,

$$U(k) = \frac{14}{15} k^4 \int_{\Lambda}^{\infty} dp \theta_{k,p,q}(t) \frac{[E(p)]^2}{p^2} \text{ for } k \rightarrow 0. \quad (19)$$

where $\theta_{k,p,q}(t) = 1/[\mu_{k,p,q}(t) + \nu_0(k^2 + p^2 + q^2)]$ and $\mu_{k,p,q}(t)$ is an “eddy-damping rate” of the third-order moments associated with the wavevectors \mathbf{k} , \mathbf{p} , and \mathbf{q} .

Thus, Lam's postulate (which was based on intuitive physical arguments) that $U(k)$ is small for small k is consistent with classical closure theory.

The advantage of the classical theory is that the energy equation is always satisfied and no restriction on the magnitude of Λ is imposed—so long as Λ is in the inertial range. On integrating (11) with respect to k for $0 < k < \Lambda$, we obtain:

$$\frac{\partial K}{\partial t} = \bar{\Pi} - \mathcal{E}. \quad (20)$$

where K is the integral of $\bar{E}(k)$ over the resolved wavenumbers, and \mathcal{E} is defined by:

$$\mathcal{E} \equiv \int_0^\Lambda T^>(k) dk = \int_0^\Lambda 2k^2 \nu_n(k) \bar{E}(k) dk. \quad (21)$$

and $\bar{\Pi}$, the resolved energy transfer term, is given by:

$$\bar{\Pi} \equiv \int_0^\Lambda \bar{T}(k) dk.$$

The *net eddy viscosity*, $\nu_n(k, t)$, is defined^{2,5,8-9} as

$$\nu_n(k) \equiv \nu_d(k) - \nu_b(k). \quad (22)$$

and $\nu_b(k, t)$, the *back-scatter viscosity*, is given by

$$\nu_b(k) \equiv U(k)/(2k^2 \bar{E}(k)). \quad (23)$$

From (14) and (23), one can show¹⁰ that for k in the inertial range and $k \ll \Lambda$, the ratio of $\nu_b(k)$ to $\nu_d(k)$ is equal to $\frac{14}{15}(k/\Lambda)^{11/3}$. Spectral large-eddy simulations (LES) of Lesieur and Rogallo^{5,11} was based on the resolvable scale Navier-Stokes equation

$$\left(\frac{\partial}{\partial t} + [\nu_0 + \nu_n(k)]k^2\right)v_\alpha^<(\mathbf{k}, t) = M_{\alpha\beta\gamma}(k) \int \int d\mathbf{p} d\mathbf{q} v_\beta^<(\mathbf{p}, t) v_\gamma^<(\mathbf{q}, t). \quad (24)$$

Lam emphasized that \mathcal{E} , the energy dissipation rate of the turbulent flow in question, must be related to the parameters of the turbulent eddies by an *ad hoc* postulate under his formulation. Lam's choice¹ is

$$\mathcal{E}_L = \lim_{\Lambda \rightarrow \infty} 2\nu_T(\Lambda) \int_0^\Lambda k^2 E(k) dk. \quad (25)$$

The large Λ limiting process in (25) is needed to ensure that the dissipation rate can be adequately evaluated using information available from the resolved modes alone. In Lam's approach, the value of Λ must be sufficiently large such that the dissipation function \mathcal{E}_L as given by (25) is independent of Λ . In physical variables, \mathcal{E}_L is defined by:

$$\mathcal{E}_L \equiv \nu_T(\Lambda) \left(\frac{\partial u_i^<}{\partial x_k} \right)^2. \quad (26)$$

The Smagorinsky result for ν_T is recovered if \mathcal{E}_L is eliminated between (26) and $\nu_T(\Lambda) = C_\nu \mathcal{E}_L^{1/3} \Lambda^{-4/3}$. In LES, the Lam requirement that Λ must be large enough is equivalent to requiring that (26), computed using data only from resolved modes, be “grid size” independent. In Lam's view, an LES calculation must exhibit a Kolmogorov spectrum using the resolved modes such that the limiting process in (25) is respected. If it does not, then the calculation would have no theoretical standing. Physically, if Λ is sufficiently large (so that \mathcal{E}_L is independent of Λ), the contribution of back scattering to the dissipation would be negligible. The random force \mathbf{f} , the *surrogate* of the \mathbf{g}^{fast} , does not appear explicitly in the final LES model of Lam and one needs only to provide a profile of $\langle \mathbf{ff} \rangle$ so as to introduce the adjustable parameter ϵ used in computing ν_T .

Conclusion

Thus, we find that Lam's formulation of ϵ -RNG¹ is essentially the physical space version of the spectral classical closure theory² with $\nu_n(k)$ being replaced by a phenomenological k -independent ν_T , but which now depends on arbitrary parameter ϵ .

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